

SPHERICAL HARMONICS

FENG DAI AND YUAN XU

ABSTRACT. This is Chapter 1 of the book *Approximation Theory and Harmonic Analysis on Spheres and Balls* by the authors. It provides a self-contained introduction to spherical harmonics. The book will be published as a title in *Springer Monographs in Mathematics* by Springer in 2013. The table of contents of the book is attached at the end of this file.

In this chapter we introduce spherical harmonics and study their properties. Most of the material of this chapter, except the last section, is classical. We strive for a succinct account of the theory of spherical harmonics. After a standard treatment of the space of spherical harmonics and orthogonal bases in the first section, the orthogonal projection operator and reproducing kernels, also known as zonal harmonics, are developed in greater details in the second section, because of their central role in harmonic analysis and approximation theory. As an application of the addition formula, it is shown in the third section that there exist bases of spherical harmonics consisting of entirely zonal harmonics. The Laplace–Beltrami operator is discussed in the fourth section, where an elementary and self-contained approach is adopted. Spherical coordinates and an explicit orthonormal basis of spherical harmonics in these coordinates are presented the fifth section. These formulas in two and three variables are collected in the sixth section for easy reference, since they are most often used in applications. The connection to group representation is treated briefly in the seventh section. The last section deals with derivatives and integrals on the sphere. With the introduction of angular derivatives that are first-order differential operators acting on the large circles of intersections of the sphere and the coordinate planes, it is shown that the Laplace–Beltrami operator can be decomposed into second-order angular derivatives. These derivative operators will play an important role in approximation theory on the sphere. They are used to derive several integral formulas on the sphere.

1. SPACE OF SPHERICAL HARMONICS AND ORTHOGONAL BASES

We begin by introducing some notation that will be used throughout this book. For $x \in \mathbb{R}^d$, we write $x = (x_1, \dots, x_d)$. The inner product of $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y \rangle := \sum_{i=1}^d x_i y_i$ and the norm of x is denoted by $\|x\| := \sqrt{\langle x, x \rangle}$. Let \mathbb{N}_0 denote the set of nonnegative integers. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, a monomial x^α is a product $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$, which has degree $|\alpha| = \alpha_1 + \dots + \alpha_d$.

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A homogeneous polynomial P of degree n is a linear combination of monomials of degree n , that is, $P(x) = \sum_{|\alpha|=n} c_\alpha x^\alpha$, where c_α are either real or complex numbers. A polynomial of (total) degree at most n is of the form $P(x) = \sum_{|\alpha| \leq n} c_\alpha x^\alpha$. Let \mathcal{P}_n^d denote the space of real homogeneous polynomials of degree n and let Π_n^d denote the space of real polynomials of degree at most n . Counting the cardinalities of $\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}$ and $\{\alpha \in \mathbb{N}_0^d : |\alpha| \leq n\}$ shows that

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{n} \quad \text{and} \quad \dim \Pi_n^d = \binom{n+d}{n}.$$

Let ∂_i denote the partial derivative in the i -th variable and Δ the Laplacian operator

$$\Delta := \partial_1^2 + \cdots + \partial_d^2.$$

Definition 1.1. For $n = 0, 1, 2, \dots$ let \mathcal{H}_n^d be the linear space of real harmonic polynomials, homogeneous of degree n , on \mathbb{R}^d , that is,

$$\mathcal{H}_n^d := \{P \in \mathcal{P}_n^d : \Delta P = 0\}.$$

Spherical harmonics are the restrictions of elements in \mathcal{H}_n^d to the unit sphere. If $Y \in \mathcal{H}_n^d$, then $Y(x) = \|x\|^n Y(x')$ where $x = \|x\|x'$ and $x' \in \mathbb{S}^{d-1}$. Strictly speaking, one should make a distinction between \mathcal{H}_n^d and its restriction to the sphere. We will, however, also call \mathcal{H}_n^d the space of spherical harmonics. When it is necessary to emphasize the restriction to the sphere, we shall use the notation $\mathcal{H}_n^d|_{\mathbb{S}^{d-1}}$. In the same vein, we shall define $\mathcal{P}_n(\mathbb{S}^{d-1}) := \mathcal{P}_n^d|_{\mathbb{S}^{d-1}}$ and $\Pi_n(\mathbb{S}^{d-1}) := \Pi_n^d|_{\mathbb{S}^{d-1}}$.

Spherical harmonics of different degrees are orthogonal with respect to

$$(1.1) \quad \langle f, g \rangle_{\mathbb{S}^{d-1}} := \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(x)g(x) d\sigma(x)$$

where $d\sigma$ is the surface area measure and ω_d denotes the surface area of \mathbb{S}^{d-1} ,

$$(1.2) \quad \omega_d := \int_{\mathbb{S}^{d-1}} d\sigma = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Theorem 1.2. If $Y_n \in \mathcal{H}_n^d$, $Y_m \in \mathcal{H}_m^d$, and $n \neq m$, then $\langle Y_n, Y_m \rangle_{\mathbb{S}^{d-1}} = 0$.

Proof. Let $\frac{\partial}{\partial r}$ denote the normal derivative. Since Y_n is homogeneous, $Y_n(x) = r^n Y_n(x')$, where $x = rx'$ and $x' \in \mathbb{S}^{d-1}$, so that $\frac{\partial Y_n}{\partial r}(x') = n Y_n(x')$ for $x' \in \mathbb{S}^{d-1}$ and $n \geq 0$. By Green's identity,

$$\begin{aligned} (n-m) \int_{\mathbb{S}^{d-1}} Y_n Y_m d\sigma &= \int_{\mathbb{S}^{d-1}} \left(Y_m \frac{\partial Y_n}{\partial r} - Y_n \frac{\partial Y_m}{\partial r} \right) d\sigma \\ &= \int_{\mathbb{B}^d} (Y_m \Delta Y_n - Y_n \Delta Y_m) dx = 0, \end{aligned}$$

since $\Delta Y_n = 0$ and $\Delta Y_m = 0$. □

Theorem 1.3. For $n = 0, 1, 2, \dots$, there is a decomposition of \mathcal{P}_n^d ,

$$(1.3) \quad \mathcal{P}_n^d = \bigoplus_{0 \leq j \leq n/2} \|x\|^{2j} \mathcal{H}_{n-2j}^d.$$

In other words, for each $P \in \mathcal{P}_n^d$, there is a unique decomposition

$$(1.4) \quad P(x) = \sum_{0 \leq j \leq n/2} \|x\|^{2j} P_{n-2j}(x) \quad \text{with} \quad P_{n-2j} \in \mathcal{H}_{n-2j}^d.$$

Proof. The proof uses induction. Evidently $\mathcal{P}_0^d = \mathcal{H}_0^d$ and $\mathcal{P}_1^d = \mathcal{H}_1^d$. Since $\Delta \mathcal{P}_n^d \subset \mathcal{P}_{n-2}^d$, $\dim \mathcal{H}_n^d \geq \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d$. Suppose the statement holds for $m = 0, 1, \dots, n-1$. Then $\|x\|^2 \mathcal{P}_{n-2}^d$ is a subspace of \mathcal{P}_n^d and it is isomorphic to \mathcal{P}_{n-2}^d . By the induction hypothesis, $\|x\|^2 \mathcal{P}_{n-2}^d = \bigoplus_{0 \leq j \leq n/2-1} \|x\|^{2j+2} \mathcal{H}_{n-2-2j}^d$. Hence, by the previous theorem, \mathcal{H}_n^d is orthogonal to $\|x\|^2 \mathcal{P}_{n-2}^d$, so that $\dim \mathcal{H}_n^d + \dim \mathcal{P}_{n-2}^d \leq \dim \mathcal{P}_n^d$. Consequently, $\mathcal{P}_n^d = \mathcal{H}_n^d \oplus \|x\|^2 \mathcal{P}_{n-2}^d$. \square

Corollary 1.4. For $n = 0, 1, 2, \dots$,

$$(1.5) \quad \dim \mathcal{H}_n^d = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d = \binom{n+d-1}{n} - \binom{n+d-3}{n-2},$$

where it is agreed that $\dim \mathcal{P}_{n-2}^d = 0$ for $n = 0, 1$.

Corollary 1.5. For $n \in \mathbb{N}$, $\Pi_n(\mathbb{S}^{d-1}) = \mathcal{P}_n(\mathbb{S}^{d-1}) \oplus \mathcal{P}_{n-1}(\mathbb{S}^{d-1})$ and

$$(1.6) \quad \dim \Pi_n(\mathbb{S}^{d-1}) = \dim \mathcal{P}_n^d + \dim \mathcal{P}_{n-1}^d = \binom{n+d-1}{n} + \binom{n+d-2}{n-1}.$$

Proof. By Theorem 1.3, $\Pi_n(\mathbb{S}^{d-1})$ can be written as a direct sum of \mathcal{H}_k^d for $0 \leq k \leq n$, which gives the stated decomposition by (1.3). Moreover,

$$\dim \Pi_n(\mathbb{S}^{d-1}) = \sum_{k=0}^n \dim \mathcal{H}_k^d = \sum_{k=0}^n (\dim \mathcal{P}_k^d - \dim \mathcal{P}_{k-2}^d)$$

by (1.5), which simplifies to (1.6). \square

The orthogonality and homogeneity define spherical harmonics.

Proposition 1.6. If P is a homogeneous polynomial of degree n and P is orthogonal to all polynomials of degree less than n with respect to $\langle \cdot, \cdot \rangle_{\mathbb{S}^{d-1}}$, then $P \in \mathcal{H}_n^d$.

Proof. Since $P \in \mathcal{P}_n^d$, P can be expressed as in (1.4). The orthogonality then shows that $P = P_n \in \mathcal{H}_n^d$. \square

Let $O(d)$ denote the orthogonal group, the group of $d \times d$ orthogonal matrices, and let $SO(d) = \{g \in O(d) : \det g = 1\}$ be the special orthogonal group. A rotation in \mathbb{R}^d is determined by an element in $SO(d)$.

Theorem 1.7. The space \mathcal{H}_n^d is invariant under the action $f(x) \mapsto f(Qx)$, $Q \in O(d)$. Moreover, if $\{Y_\alpha\}$ is an orthonormal basis of \mathcal{H}_n^d , then so is $\{Y_\alpha(Q\{\cdot\})\}$.

Proof. Since Δ is invariant under the rotation group $O(d)$ (writing $\Delta = \nabla \cdot \nabla$ and changing variables), if $Y \in \mathcal{H}_n^d$ and $Q \in O(d)$ then $Y(Qx) \in \mathcal{H}_n^d$. That $\{Y_\alpha(Qx)\}$ is an orthonormal basis of \mathcal{H}_n^d whenever $\{Y_\alpha(x)\}$ follows from

$$\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Y_\alpha(Qx) Y_\beta(Qx) d\sigma(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Y_\alpha(x) Y_\beta(x) d\sigma(x) = \delta_{\alpha,\beta},$$

which holds under a change of variables since $d\sigma$ is invariant under $O(d)$. \square

Besides $\langle f, g \rangle_{\mathbb{S}^{d-1}}$, another useful inner product can be defined on \mathcal{P}_n^d through the action of differentiation. For $\alpha \in \mathbb{N}_0^d$, let $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$. Let $(a)_n := a(a+1) \dots (a+n-1)$ be the Pochhammer symbol.

Theorem 1.8. For $p, q \in \mathcal{P}_n^d$, define a bilinear form

$$(1.7) \quad \langle p, q \rangle_\partial := p(\partial)q,$$

where $p(\partial)$ is the differential operator defined by replacing x^α in $p(x)$ by ∂^α . Then

1. $\langle p, q \rangle_\partial$ is an inner product on \mathcal{P}_n^d ;
2. the reproducing kernel of this inner product is $k_n(x, y) := \langle x, y \rangle^n / n!$; that is,

$$\langle k_n(x, \cdot), p \rangle_\partial = p(x), \quad \forall p \in \mathcal{P}_n^d;$$

3. for $p \in \mathcal{P}_n^d$ and $q \in \mathcal{H}_n^d$,

$$\langle p, q \rangle_\partial = 2^n \left(\frac{d}{2} \right)_n \langle p, q \rangle_{\mathbb{S}^{d-1}}.$$

Proof. Let $p, q \in \mathcal{P}_n^d$ be given by $p(x) = \sum_{|\alpha|=n} a_\alpha x^\alpha$ and $q(x) = \sum_{|\alpha|=n} b_\alpha x^\alpha$, where $a_\alpha, b_\alpha \in \mathbb{R}$. Then,

$$(1.8) \quad \langle p, q \rangle_\partial = \sum_{|\alpha|=n} a_\alpha \partial^\alpha \sum_{|\beta|=n} b_\beta x^\beta = \sum_{|\alpha|=n} \alpha! a_\alpha b_\alpha,$$

which implies, in particular, that $\langle p, p \rangle_\partial > 0$ for $p \neq 0$. It follows then that $\langle \cdot, \cdot \rangle_\partial$ is an inner product on \mathcal{P}_n^d . By the multinomial formula, for $q_\alpha(x) = x^\alpha$, $|\alpha| = n$,

$$\langle k_n(x, \cdot), q_\alpha \rangle_\partial = \frac{1}{n!} \sum_{|\beta|=n} \binom{n}{\beta} x^\beta \frac{\partial^\beta}{\partial y^\beta} y^\alpha = q_\alpha(x),$$

which shows that $k_n(x, y)$ is the reproducing kernel with respect to $\langle \cdot, \cdot \rangle_\partial$.

We now prove item 3. Integrating by parts shows that

$$\int_{\mathbb{R}^d} \partial_i f(x) g(x) e^{-\|x\|^2/2} dx = - \int_{\mathbb{R}^d} f(x) (\partial_i g(x) - x_i g(x)) e^{-\|x\|^2/2} dx.$$

Since $p(\partial)q$ is a constant, using this integration by parts repeatedly shows that

$$\begin{aligned} \langle p, q \rangle_\partial &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} p(\partial) q(x) e^{-\|x\|^2/2} dx \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} q(x) (p(x) + s(x)) e^{-\|x\|^2/2} dx \end{aligned}$$

where $s \in \Pi_{n-1}^d$. Since $q \in \mathcal{H}_n^d$ and $p \in \mathcal{P}_n^d$, switching to polar integral and using the orthogonality of \mathcal{H}_n^d , we obtain

$$\langle p, q \rangle_\partial = \frac{1}{(2\pi)^{d/2}} \int_0^\infty r^{2n+d-1} e^{-r^2/2} dr \int_{\mathbb{S}^{d-1}} q(x') p(x') d\sigma(x').$$

Evaluating the integral in r and simplifying by (1.2) concludes the proof. \square

A large number of spherical harmonic polynomials can be defined explicitly through differentiation. Let us denote the standard basis of \mathbb{R}^d by

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0) \dots, e_d = (0, \dots, 0, 1).$$

Theorem 1.9. *Let $d > 2$. For $\alpha \in \mathbb{N}_0^d$, $n = |\alpha|$, define*

$$(1.9) \quad p_\alpha(x) := \frac{(-1)^n}{2^n \left(\frac{d-2}{2} \right)_n} \|x\|^{2|\alpha|+d-2} \partial^\alpha \{ \|x\|^{-d+2} \}.$$

Then

1. $p_\alpha \in \mathcal{H}_n^d$ and p_α is the monic spherical harmonic of the form

$$(1.10) \quad p_\alpha(x) = x^\alpha + \|x\|^2 q_\alpha(x), \quad q_\alpha \in \mathcal{P}_{n-2}^d.$$

2. p_α satisfies the recurrence relation

$$(1.11) \quad p_{\alpha+e_i}(x) = x_i p_\alpha(x) - \frac{1}{2n+d-2} \|x\|^2 \partial_i p_\alpha(x),$$

3. $\{p_\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$ is a basis of \mathcal{H}_n^d .

Proof. Taking the derivative of $p_\alpha(x)$ gives immediately the recurrence relation (1.11). Clearly $p_0(x) = 1$. By induction, the recurrence relation shows that p_α is a homogeneous polynomial of degree n and it is of the form (1.10). We now show that p_α is a spherical harmonic. For $g \in \mathcal{P}_n^d$ and $\rho \in \mathbb{R}$, a quick computation using $\sum_{i=1}^d x_i \partial_i g(x) = n g(x)$ shows that

$$(1.12) \quad \Delta(\|x\|^\rho g) = \rho(2n + \rho + d - 2) \|x\|^{\rho-2} g + \|x\|^\rho \Delta g.$$

In particular, setting $n = 0$ and $g(x) = 1$ gives $\Delta(\|x\|^{-d+2}) = 0$. Furthermore, setting $g = p_\alpha$ and $\rho = -2n - d + 2$ in (1.12) leads to

$$\Delta p_\alpha(x) = \frac{(-1)^n}{2^n (d/2 - 1)_n} \|x\|^{2|\alpha|+d-2} \partial^\alpha \Delta \{\|x\|^{-d+2}\} = 0.$$

Thus, $p_\alpha \in \mathcal{H}_n^d$. Since $\|x\|^2 q(x)$ is a linear combination of the monomials x^β with $\beta_d \geq 2$, by (1.10) and the linear independence of $\{x^\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$, it follows that the elements in the set $\{p_\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$ are linearly independent. The cardinality of the set is

$$\dim \mathcal{P}_n^{d-1} + \dim \mathcal{P}_{n-1}^{d-1} = \binom{n+d-2}{d-2} + \binom{n+d-3}{d-2},$$

which is, by a simple identity of binomial coefficients and (1.5), precisely $\dim \mathcal{H}_n^d$. This completes the proof. \square

The right-hand side of (1.9) is called Maxwell's representation of harmonic polynomials ([8, 10]). The complete set of $\{p_\alpha : |\alpha| = n\}$ is necessarily linearly dependent by its cardinality. Moreover, by (1.9),

$$p_{\alpha+2e_1} + \dots + p_{\alpha+2e_d} = \frac{(-1)^n}{2^n (\frac{d}{2})_n} \|x\|^{2|\alpha|+d-2} \partial^\alpha \Delta \{\|x\|^{-d+2}\} = 0,$$

which gives $\dim \mathcal{P}_{n-2}^d$ linearly dependent relations among $\{p_\alpha : |\alpha| = n\}$. The set $\{p_\alpha : |\alpha| = n\}$ evidently contains many bases of \mathcal{H}_n^d . The basis in item three of Theorem 1.9 is but one convenient choice. The proof of Theorem 1.9 relies on the fact that $\|x\|^{-d+2}$ is a harmonic function in $\mathbb{R}^d \setminus \{0\}$ for $d > 2$. In the case of $d = 2$, we need to replace this function by $\log \|x\|$. Since the case $d = 2$ corresponds to the classical Fourier series, we leave the analogue of Theorem 1.9 for $d = 2$ to the interested reader.

The basis $\{p_\alpha : |\alpha| = n, \alpha_d = 0 \text{ or } 1\}$ of \mathcal{H}_n^d is not orthonormal. In fact, the elements of this basis are not mutually orthogonal. Orthonormal bases can be constructed by applying the Gram-Schmidt process. An explicit orthonormal basis for \mathcal{H}_n^d will be given in Section 5 in terms of spherical coordinates.

2. PROJECTION OPERATORS AND ZONAL HARMONICS

Let $L^2(\mathbb{S}^{d-1})$ denote the space of square integrable functions on \mathbb{S}^{d-1} . Let

$$\text{proj}_n : L^2(\mathbb{S}^{d-1}) \mapsto \mathcal{H}_n^d$$

denote the orthogonal projection from $L^2(\mathbb{S}^{d-1})$ onto \mathcal{H}_n^d . If $P \in \mathcal{P}_n^d$ then $P = P_n + \|x\|^2 Q_n$, where $P_n \in \mathcal{H}_n^d$ and $Q_n \in \mathcal{P}_{n-2}^d$, by (1.4), so that $\text{proj}_n P = P_n$. In particular, (1.11) shows that p_α defined in (1.9) is the orthogonal projection of the function $q_\alpha(x) = x^\alpha$; that is, $p_\alpha = \text{proj}_n q_\alpha$. This leads to the following:

Lemma 2.1. *Let $p \in \mathcal{P}_n^d$. Then*

$$(2.1) \quad \text{proj}_n p = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{4^j j! (-n+2-d/2)_j} \|x\|^{2j} \Delta^j p.$$

Proof. By linearity, it suffices to consider p being $q_\alpha(x) = x^\alpha$. By Theorem 1.9, $\text{proj}_n q_\alpha(x) = p_\alpha(x)$, and the proof amounts to showing that $p_\alpha(x)$ defined in (1.9) can be expanded as in (2.1). We use induction on n . The case $n = 0$ is evident. Assume that (2.1) has been established for $m = 0, 1, \dots, n$. Applying (2.1) to $q_\alpha(x)$, $|\alpha| = n$, it follows that

$$\begin{aligned} \partial^\alpha \{ \|x\|^{-d+2} \} &= (-1)^n 2^n \left(\frac{d}{2} - 1 \right)_n \|x\|^{-2n-d+2} \\ &\quad \times \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{4^j j! (-n+2-d/2)_j} \|x\|^{2j} \Delta^j \{ x^\alpha \}. \end{aligned}$$

Applying ∂_i to this identity, we obtain

$$\begin{aligned} \partial_i \partial^\alpha \{ \|x\|^{-d+2} \} &= (-1)^n 2^n \left(\frac{d}{2} - 1 \right)_n (-2n-d+2) \|x\|^{-2n-d+2} \\ &\quad \times \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \frac{1}{4^j j! (-n+1-d/2)_j} \|x\|^{2j} [x_i \Delta^j \{ x^\alpha \} + 2j \Delta^{j-1} \partial_i \{ x^\alpha \}]. \end{aligned}$$

The terms in the square brackets are exactly $\Delta^j \{ x_i x^\alpha \}$ and the constant in front simplifies to $(-1)^{n+1} 2^{n+1} \left(\frac{d}{2} - 1 \right)_{n+1}$, so that the equation (2.1) holds for $p(x) = x_i x^\alpha$. This completes the induction. \square

Definition 2.2. *The reproducing kernel $Z_n(\cdot, \cdot)$ of \mathcal{H}_n^d is uniquely determined by*

$$(2.2) \quad \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Z_n(x, y) p(y) d\sigma(y) = p(x), \quad \forall p \in \mathcal{H}_n^d, \quad x \in \mathbb{S}^{d-1}$$

and the requirement that $Z_n(x, \cdot)$ be an element of \mathcal{H}_n^d for each fixed x .

That the kernel is well defined and unique follows from the Riesz representation theorem applied to the linear functional $L(Y) := Y(x)$, $Y \in \mathcal{H}_n^d$, for a fixed $x \in \mathbb{S}^{d-1}$.

Lemma 2.3. *In terms of an orthonormal basis $\{Y_j : 1 \leq j \leq \dim \mathcal{H}_n^d\}$ of \mathcal{H}_n^d ,*

$$(2.3) \quad Z_n(x, y) = \sum_{k=1}^{\dim \mathcal{H}_n^d} Y_k(x) Y_k(y), \quad x, y \in \mathbb{S}^{d-1},$$

and, despite (2.3), Z_n is independent of the particular choice of basis of \mathcal{H}_n^d .

Proof. Since $Z_n(x, \cdot) \in \mathcal{H}_n^d$, it can be expressed as $Z_n(x, y) = \sum_k c_k Y_k(y)$ where the coefficients are determined by (2.2) as $c_k = Y_k(x)$. The uniqueness implies that Z_n is independent of the choice of basis. This can also be shown directly as follows. Let $\mathbb{Y}_n = (Y_1, \dots, Y_N)$ with $N = \dim \mathcal{H}_n^d$ and regard it as a column vector. Then $Z_n(x, y) = [\mathbb{Y}_n(x)]^{\text{tr}} \mathbb{Y}_n(y)$. If $\{Y'_j : 1 \leq j \leq N\}$ is another orthonormal basis of \mathcal{H}_n^d , then $\mathbb{Y}'_n = Q \mathbb{Y}_n$. Since the orthonormality of $\{Y_j\}$ can be expressed as the fact that $\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \mathbb{Y}_n(x) [\mathbb{Y}_n(x)]^{\text{tr}} d\sigma(x)$ is an identity matrix, it follows readily that Q is an orthogonal matrix. Hence, $Z_n(x, y) = [\mathbb{Y}_n(x)]^{\text{tr}} Q^{\text{tr}} Q \mathbb{Y}_n(y) = [\mathbb{Y}'_n(x)]^{\text{tr}} \mathbb{Y}'_n(y)$. \square

The reproducing kernel is also the kernel for the projection operator.

Lemma 2.4. *The projection operator can be written as*

$$(2.4) \quad \text{proj}_n f(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) Z_n(x, y) d\sigma(y).$$

Proof. Since $\text{proj}_n f \in \mathcal{H}_n^d$, it can be expanded in terms of the orthonormal basis $\{Y_j, 1 \leq j \leq N_n\}$, $N_n = \dim \mathcal{H}_n^d$, of \mathcal{H}_n^d , where the coefficients are determined by the orthonormality,

$$\text{proj}_n f(x) = \sum_{j=1}^{N_n} c_j Y_j(x) \quad \text{with} \quad c_j = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) Y_j(y) d\sigma(y).$$

If we pull out the integral in front of the sum, this is (2.4) by (2.3). \square

Lemma 2.5. *The kernel $Z_n(\cdot, \cdot)$ satisfies the following properties:*

1. *For every $\xi, \eta \in \mathbb{S}^{d-1}$,*

$$(2.5) \quad \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Z_n(\xi, y) Z_n(\eta, y) d\sigma(y) = Z_n(\xi, \eta).$$

2. *$Z_n(x, y)$ depends only on $\langle x, y \rangle$.*

Proof. By Corollary 1.7, the uniqueness of $Z_n(x, y)$ shows that $Z_n(Qx, Qy) = Z_n(x, y)$ for all $Q \in O(d)$. Since for $x, y \in \mathbb{S}^{d-1}$ there exists a $Q \in SO(d)$ such that $Qx = (0, \dots, 0, 1)$ and $Qy = (0, \dots, 0, \sqrt{1 - \langle x, y \rangle^2}, \langle x, y \rangle)$, this shows that $Z_n(x, y)$ depends only on $\langle x, y \rangle$. \square

From the second property of the lemma, $Z_n(x, y) = F_n(\langle x, y \rangle)$, which is often called a zonal harmonic, since it is harmonic and depends only on $\langle x, y \rangle$. We now derive a closed formula for F_n , which turns out to be a multiple of the Gegenbauer polynomial, C_n^λ , of degree n defined, for $\lambda > 0$ and $n \in \mathbb{N}_0$, by

$$(2.6) \quad C_n^\lambda(x) := \frac{(\lambda)_n 2^n}{n!} x^n {}_2F_1 \left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{x^2} \right),$$

where ${}_2F_1$ is the hypergeometric function. The properties of the Gegenbauer polynomials are collected in Appendix B.

Theorem 2.6. *For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{S}^{d-1}$, $d \geq 3$,*

$$(2.7) \quad Z_n(x, y) = \frac{n + \lambda}{\lambda} C_n^\lambda(\langle x, y \rangle), \quad \lambda = \frac{d-2}{2}.$$

Proof. Let $p \in \mathcal{H}_n^d$. By Theorem 1.8, $p(x) = \langle k_n(x, \cdot), p \rangle_{\partial}$. For fixed x , it follows from the same theorem that

$$\begin{aligned} p(x) &= \langle k_n(x, \cdot), p \rangle_{\partial} = \langle \text{proj}_n(k_n(x, \cdot)), p \rangle_{\partial} \\ &= \frac{2^n (d/2)_n}{\omega_d} \int_{\mathbb{S}^{d-1}} \text{proj}_n[k_n(x, \cdot)](y) p(y) d\sigma(y). \end{aligned}$$

Since the kernel $Z_n(\cdot, \cdot)$ is uniquely determined by the reproducing property, this shows that $Z_n(x, y) = 2^n (\frac{d}{2})_n \text{proj}_n[k_n(x, \cdot)](y)$. Since $k_n(x, \cdot)$ is a homogeneous polynomial of degree n and, taking the derivative on y , we have $\Delta^j k_n(x, y) = \|x\|^{2j} k_{n-2j}(x, y)$, as is easily seen from $\partial_i k_n(x, y) = x_i k_{n-1}(x, y)$, Lemma 2.1 shows, for $x, y \in \mathbb{S}^{d-1}$, that

$$Z_n(x, y) = 2^n \left(\frac{d}{2}\right)_n \text{proj}_n[k_n(x, \cdot)](y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(\frac{d}{2})_n 2^{n-2j}}{j!(1-n-\lambda)_j} k_{n-2j}(x, y).$$

Using the fact $1/(n-2j)! = (-n)_{2j}/n! = 2^{2j}(-\frac{n}{2})_j(-\frac{n+1}{2})_j/n!$, we conclude then

$$\begin{aligned} Z_n(x, y) &= \frac{n+\lambda}{\lambda} \frac{(\lambda)_n 2^n}{n!} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-\frac{n}{2})_j (-\frac{n+1}{2})_j}{j!(1-n-\lambda)_j} \langle x, y \rangle^{n-2j} \\ &= \frac{n+\lambda}{\lambda} \frac{(\lambda)_n 2^n}{n!} \langle x, y \rangle^n {}_2F_1 \left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 1-n-\lambda \end{matrix}; \frac{1}{\langle x, y \rangle^2} \right), \end{aligned}$$

from which the stated result follows from (2.6). \square

Let $\{Y_i : 1 \leq i \leq \dim \mathcal{H}_n^d\}$ be an orthonormal basis of \mathcal{H}_n^d . Then (2.7) states that

$$(2.8) \quad \sum_{j=1}^{\dim \mathcal{H}_n^d} Y_j(x) Y_j(y) = \frac{n+\lambda}{\lambda} C_n^\lambda(\langle x, y \rangle), \quad \lambda = \frac{d-2}{2}.$$

This identity is usually referred to as the addition formula of spherical harmonics, since for $d = 2$ it is the addition formula of the cosine function (see Section 6).

Corollary 2.7. For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{S}^{d-1}$, $d \geq 3$,

$$(2.9) \quad |Z_n(x, y)| \leq \dim \mathcal{H}_n^d \quad \text{and} \quad Z_n(x, x) = \dim \mathcal{H}_n^d.$$

Proof. Set $F_n(t) := \frac{n+\lambda}{\lambda} C_n^\lambda(t)$. By (2.7), $Z_n(x, x) = F_n(1)$ is a constant for all $x \in \mathbb{S}^{d-1}$. Setting $x = y$ in (2.3) and integrating over \mathbb{S}^{d-1} , we obtain

$$F_n(1) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} Z_n(\langle x, x \rangle) d\sigma(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \sum_{k=1}^{\dim \mathcal{H}_n^d} Y_k^2(x) d\sigma(x) = \dim \mathcal{H}_n^d.$$

The inequality follows from applying the Cauchy-Schwarz inequality to (2.3). \square

Because of the relation (2.7), the Gegenbauer polynomials with $\lambda = \frac{d-2}{2}$ are also called ultraspherical polynomials. A number of properties of the Gegenbauer polynomials can be obtained from the zonal spherical harmonics. For example, the corollary implies that $C_n^\lambda(1) = \frac{\lambda}{n+\lambda} \dim \mathcal{H}_n^d$. Here is another example:

Corollary 2.8. For $\lambda = \frac{d-2}{2}$, the Gegenbauer polynomials C_n^λ satisfy the orthogonality relation

$$(2.10) \quad \frac{\omega_{d-1}}{\omega_d} \int_{-1}^1 C_n^\lambda(t) C_m^\lambda(t) (1-t^2)^{\lambda-\frac{1}{2}} dt = h_n^\lambda \delta_{m,n},$$

where

$$h_n^\lambda = \frac{\lambda}{n+\lambda} C_n^\lambda(1).$$

Proof. Set again $F_n(t) = \frac{n+\lambda}{\lambda} C_n^\lambda(t)$. Since $Z_n(x, \cdot)$ and $Z_m(x, \cdot)$ are orthogonal over \mathbb{S}^{d-1} , and by (A.5.1), their integrals can be written as an integral of one variable, we obtain

$$\begin{aligned} \frac{\omega_{d-1}}{\omega_d} \int_{-1}^1 F_n(t) F_m(t) (1-t^2)^{\frac{d-3}{2}} dt &= \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} F_n(\langle x, y \rangle) F_m(\langle x, y \rangle) d\sigma(y) \\ &= F_n(1) \delta_{m,n} = (\dim \mathcal{H}_n^d) \delta_{m,n}, \end{aligned}$$

where the second line follows from (2.5) and Corollary 2.7. \square

The functions on \mathbb{S}^{d-1} that depend only on $\langle x, y \rangle$ are analogues of radial functions on \mathbb{R}^d . For such functions, there is a Funk-Hecke formula given below.

Theorem 2.9. Let f be an integrable function such that $\int_{-1}^1 |f(t)| (1-t^2)^{(d-3)/2} dt$ is finite and $d \geq 2$. Then for every $Y_n \in \mathcal{H}_n^d$,

$$(2.11) \quad \int_{\mathbb{S}^{d-1}} f(\langle x, y \rangle) Y_n(y) d\sigma(y) = \lambda_n(f) Y_n(x), \quad x \in \mathbb{S}^{d-1},$$

where $\lambda_n(f)$ is a constant defined by

$$\lambda_n(f) = \omega_{d-1} \int_{-1}^1 f(t) \frac{C_n^{\frac{d-2}{2}}(t)}{C_n^{\frac{d-2}{2}}(1)} (1-t^2)^{\frac{d-3}{2}} dt.$$

Proof. If f is a polynomial of degree m , then we can expand f in terms of the Gegenbauer polynomials

$$f(t) = \sum_{k=0}^m \lambda_k \frac{k + \frac{d-2}{2}}{\frac{d-2}{2}} C_k^{\frac{d-2}{2}}(t).$$

where λ_k are determined by the orthogonality of the Gegenbauer polynomials,

$$\lambda_k = \frac{c_d}{C_k^{\frac{d-2}{2}}(1)} \int_{-1}^1 f(t) C_k^{\frac{d-2}{2}}(t) (1-t^2)^{\frac{d-3}{2}} dt,$$

and $c_d^{-1} = \int_{-1}^1 (1-t^2)^{\frac{d-3}{2}} dt = \omega_d / \omega_{d-1}$. From (2.7) and the reproducing property of $Z_n(x, y)$, it follows that for $n \leq m$

$$\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(\langle x, y \rangle) Y_n(y) d\sigma(y) = \lambda_n Y_n(x), \quad x \in \mathbb{S}^{d-1}.$$

Since $\lambda_n / \omega_d = \lambda_n(f)$ by definition, we have established the Funk-Hecke formula (2.11) for polynomials, and hence, by the Weierstrass theorem, for continuous functions, and the function satisfying the integrable condition in the statement can be approximated by a sequence of continuous functions. \square

3. ZONAL BASIS OF SPHERICAL HARMONICS

In view of the addition formula of spherical harmonics, one may ask whether there is a basis of spherical harmonics that consists entirely of zonal harmonics. This question is closely related to the problem of interpolation on the sphere.

Throughout this subsection, we fix n , set $N = \dim \mathcal{H}_n^d$ and fix $\{Y_1, \dots, Y_N\}$ as an orthonormal basis for \mathcal{H}_n^d . Let $\{x_1, \dots, x_N\}$ be a collection of points on \mathbb{S}^{d-1} . We let $M_1 := Y_1(x_1)$ and, for $k = 2, 3, \dots, N$, define matrices

$$M_k := \begin{bmatrix} Y_1(x_1) & \dots & Y_1(x_k) \\ \vdots & \dots & \vdots \\ Y_k(x_1) & \dots & Y_k(x_k) \end{bmatrix}, \quad M_k(x) := \left[\begin{array}{c|c} & Y_1(x) \\ & \vdots \\ M_{k-1} & Y_{k-1}(x) \\ \hline Y_k(x_1), \dots, Y_k(x_{k-1}) & Y_k(x) \end{array} \right].$$

The product of M_N and its transpose M_N^T can be summed on applying the addition formula (2.8), as $M_N^T M_N = [Z_n(x_i, x_j)]_{i,j=1}^N$, which shows, in particular,

$$(3.1) \quad \det[Z_n(x_i, x_j)]_{i,j=1}^N = (\det M_N)^2 \geq 0.$$

This motivates the following definition.

Definition 3.1. A collection of points $\{x_1, \dots, x_N\}$ in \mathbb{S}^{d-1} is called a *fundamental system of degree n on the sphere \mathbb{S}^{d-1}* if

$$\det [C_n^\lambda(\langle x_i, x_j \rangle)]_{i,j=1}^N > 0, \quad \lambda = \frac{d-2}{2}.$$

Lemma 3.2. There exists a fundamental system of degree n on the sphere.

Proof. The existence of a fundamental system follows from the linear independence of $\{Y_1, \dots, Y_N\}$. Indeed, we can clearly choose $x_1 \in \mathbb{S}^{d-1}$ such that $\det M_1 = Y_1(x_1) \neq 0$. Assume that x_1, \dots, x_k , $1 \leq k \leq N-1$, have been chosen such that $\det M_k \neq 0$. The determinant $\det M_{k+1}(x)$ is a polynomial of x and cannot be identically zero by the linear independence of $\{Y_1, \dots, Y_{k+1}\}$, so that there is a $x_{k+1} \in \mathbb{S}^{d-1}$ such that $\det M_{k+1} = \det M_k(x_{k+1}) \neq 0$. In this way, we end up with a collection of points $\{x_1, \dots, x_N\}$ on \mathbb{S}^{d-1} that satisfies $\det M_N \neq 0$, which implies $\det [C_n^\lambda(\langle x_i, x_j \rangle)]_{i,j=1}^N = c_N (\det M_N)^2 > 0$, where $c_N = \lambda^N / (n + \lambda)^N$. \square

The proof shows, in fact, that there are infinitely many fundamental systems. Indeed, regarding x_1, \dots, x_N as variables, we say that $\det M_N$ is a $(d-1)N$ -dimensional polynomial in these variables, and its zero set is an algebraic surface of $\mathbb{R}^{(d-1)N}$, which necessarily has measure zero.

Theorem 3.3. If $\{x_1, \dots, x_N\}$ is a fundamental system of points on the sphere, then $\{C_n^\lambda(\langle \cdot, x_i \rangle) : i = 1, 2, \dots, N\}$, $\lambda = \frac{d-2}{2}$, is a basis of $\mathcal{H}_n^d|_{\mathbb{S}^{d-1}}$.

Proof. Let $P_i(x) = \frac{n+\lambda}{\lambda} C_n^\lambda(\langle \cdot, x_i \rangle)$. The addition theorem (2.8) gives

$$P_i(x) = \sum_{k=1}^N Y_k(x_i) Y_k(x), \quad i = 1, 2, \dots, N,$$

which shows that $\{P_1, \dots, P_N\}$ is expressed in the basis $\{Y_1, \dots, Y_N\}$ with transition matrix given by $M_N = [Y_k(x_i)]_{k,i=1}^N$. Since $\{x_1, \dots, x_N\}$ is fundamental, the matrix is invertible, by (3.1). We can then invert the system to express Y_k as a linear combination of P_1, \dots, P_N , which completes the proof. \square

A word of caution is in order. The polynomial $C_n^\lambda(\langle x, x_i \rangle)$ is, for $x \in \mathbb{S}^{d-1}$, a linear combination of the spherical harmonics according to the addition formula. It is not, however, a homogeneous polynomial of degree n in $x \in \mathbb{R}^d$; rather, it is the restriction of the homogeneous polynomial $\|x\|^n C_n^\lambda(\langle x/\|x\|, y \rangle)$ to the sphere. This is a situation in which the distinction between \mathcal{H}_n^d and $\mathcal{H}_n^d|_{\mathbb{S}^{d-1}}$ is called for; see the discussion below Definition 1.1.

Fundamental sets of points are closely related to the problem of interpolation. Indeed, it can be stated as follows: for a given set of data $\{(x_j, y_j) : 1 \leq j \leq N\}$, $x_j \in \mathbb{S}^{d-1}$ and $y_j \in \mathbb{R}$, there is a unique element $Y \in \mathcal{H}_n^d$ such that $Y(x_j) = y_j$, $j = 1, \dots, N$, if and only if the points $\{x_1, \dots, x_N\}$ form a fundamental system on the sphere. Much more interesting and challenging is the problem of choosing points in such a way that the resulting basis of zonal spherical harmonics has a relatively simple structure.

A related result is a zonal basis for the space \mathcal{P}_n of homogeneous polynomials of degree n and the space Π_n^d of all polynomials of degree at most n .

Theorem 3.4. *There exist points $\xi_{j,n} \in \mathbb{S}^{d-1}$, $1 \leq j \leq r_n^d := \dim \mathcal{P}_n^d$, such that*

- (i) *$\{\langle x, \xi_{j,n} \rangle^n : 1 \leq j \leq r_n^d\}$ is a basis for \mathcal{P}_n^d of degree n .*
- (ii) *For each polynomial $f \in \Pi_n^d$, there exist polynomials $p_j : [-1, 1] \mapsto \mathbb{R}$ for $1 \leq j \leq r_n^d$ such that*

$$f(x) = \sum_{j=1}^{r_n^d} p_j(\langle x, \xi_{j,n} \rangle).$$

Proof. Following the proof of the existence of fundamental system of points, it is easy to see that there exist points $\xi_{j,n}$ such that $f(\xi_{j,n}) = 0$ if and only if $f = 0$ for all $f \in \mathcal{P}_n^d$. For the proof of (i), we first deduce by the binomial formula that

$$f_{j,n}(x) := \langle x, \xi_{j,n} \rangle^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!} \xi_{j,n}^\alpha x^\alpha, \quad 1 \leq j \leq r_n^d.$$

Let $f \in \mathcal{P}_n^d$. Then $f(x) = \sum_{|\alpha|=n} a_\alpha x^\alpha$. By (1.8),

$$\langle f, f_{j,n} \rangle_\partial = n! \sum_{|\alpha|=n} a_\alpha \xi_{j,n}^\alpha = n! f(\xi_{j,n}),$$

which implies, by the choice of $\xi_{j,n}$, that $\langle f, f_{j,n} \rangle_\partial = 0$, $1 \leq j \leq r_n^d$, if and only if $f = 0$. Thus, $\{f_{j,n}\}^\perp = \{0\}$. Since $f_{j,n} \in \mathcal{P}_n^d$. This proves (i).

For the proof of (ii), let m be an integer, $0 \leq m \leq n-1$, and let $f_{j,m,n} := \langle x, \xi_{j,n} \rangle^m$. For $f \in \mathcal{P}_m^d$, it follows as above that $\langle f, f_{j,m,n} \rangle_\partial = m! f(\xi_{j,m})$, $1 \leq j \leq r_n^d$. Hence, if $\langle f, f_{j,m,n} \rangle_\partial = 0$ for $1 \leq j \leq r_n^d$, then $f(\xi_{j,m}) = (fg)(\xi_{j,m}) = 0$ for $1 \leq j \leq r_n^d$ and all $g \in \mathcal{P}_{n-m}^d$, which implies by the choice of $\xi_{j,n}$ and the fact that $fg \in \mathcal{P}_n^d$, that $fg = 0$ or $f = 0$. Consequently, $\mathcal{P}_m^d = \text{span}\{f_{j,m,n} : 0 \leq j \leq r_n^d\}$ for $0 \leq m \leq n$. Since $\Pi_n^d = \sum_{m=0}^n \mathcal{P}_m^d$, this proves (ii). \square

4. LAPLACE-BELTRAMI OPERATOR

The operator in the section heading is the spherical part of the Laplace operator, which we denote by Δ_0 . The operator Δ_0 plays an important role for analysis on the sphere. The usual approach to deriving this operator relies on an expression of the Laplace operator under a change of variables, which we describe first.

For $x \in \mathbb{R}^d$ let $x \mapsto u = u(x)$ be a change of variables that is a bijection, so that we can also write $x = x(u)$. Introduce the tensors

$$g_{i,j} := \sum_{k=1}^d \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} \quad \text{and} \quad g^{i,j} := \sum_{k=1}^d \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}, \quad 1 \leq i, j \leq d,$$

and let $g := \det(g_{i,j})_{i,j=1}^d$. Then $(g_{i,j})^{-1} = (g^{i,j})$. A general result in Riemannian geometry, or a bit of tensor analysis, shows that the Laplace operator satisfies

$$(4.1) \quad \Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} = \frac{1}{\sqrt{g}} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial}{\partial u_i} \sqrt{g} g^{i,j} \frac{\partial}{\partial u_j}.$$

The Laplace–Beltrami operator, i.e., the spherical part of the Laplace operator, can then be derived from (4.1) by the change of variables $x \mapsto (r, \xi_1, \dots, \xi_{d-1})$, where $r > 0$ and $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{S}^{d-1}$. For this approach and a derivation of (4.1), see [10]. We shall adopt an approach that is elementary and self-contained.

Lemma 4.1. *In the spherical–polar coordinates $x = r\xi$, $r > 0$, $\xi \in \mathbb{S}^{d-1}$, the Laplace operator satisfies*

$$(4.2) \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_0,$$

where

$$(4.3) \quad \Delta_0 = \sum_{i=1}^{d-1} \frac{\partial^2}{\partial \xi_i^2} - \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \xi_i \xi_j \frac{\partial^2}{\partial \xi_i \partial \xi_j} - (d-1) \sum_{i=1}^{d-1} \xi_i \frac{\partial}{\partial \xi_i}.$$

Proof. Since $\xi \in \mathbb{S}^{d-1}$, we have $\xi_1^2 + \dots + \xi_d^2 = 1$. We evaluate the Laplacian Δ under a change of variables $(x_1, \dots, x_d) \mapsto (r, \xi_1, \dots, \xi_{d-1})$ under $x = r\xi$, which has inverse $\xi_1 = x_1/\|x\|, \dots, \xi_{d-1} = x_{d-1}/\|x\|, r = \|x\|$. The chain rule leads to

$$(4.4) \quad \begin{aligned} \frac{\partial}{\partial x_i} &= \frac{1}{r} \frac{\partial}{\partial \xi_i} - \frac{\xi_i}{r} \sum_{j=1}^{d-1} \xi_j \frac{\partial}{\partial \xi_j} + \xi_i \frac{\partial}{\partial r}, \quad 1 \leq i \leq d-1, \\ \frac{\partial}{\partial x_d} &= -\frac{x_d}{r^2} \sum_{j=1}^{d-1} \xi_j \frac{\partial}{\partial \xi_j} + \frac{x_d}{r} \frac{\partial}{\partial r}. \end{aligned}$$

If we apply the product rule for the partial derivative on x_d , it follows that

$$\begin{aligned} \Delta &= \sum_{i=1}^{d-1} \left(\frac{1}{r} \frac{\partial}{\partial \xi_i} - \frac{\xi_i}{r} \sum_{j=1}^{d-1} \xi_j \frac{\partial}{\partial \xi_j} + \xi_i \frac{\partial}{\partial r} \right)^2 \\ &\quad + \xi_d^2 \left(-\frac{1}{r} \sum_{j=1}^{d-1} \xi_j \frac{\partial}{\partial \xi_j} + \frac{\partial}{\partial r} \right)^2 + (1 - \xi_d^2) \left(-\frac{1}{r^2} \sum_{j=1}^{d-1} \xi_j \frac{\partial}{\partial \xi_j} + \frac{1}{r} \frac{\partial}{\partial r} \right), \end{aligned}$$

where we have used $x_d = r\xi_d$, from which a straightforward, though tedious, computation, and simplification using $\xi_1^2 + \dots + \xi_d^2 = 1$, establishes (4.2) and (4.3). \square

The Laplace–Beltrami operator also satisfies a recurrence relation that can be used to derive an explicit formula for Δ_0 under a given coordinate system of \mathbb{S}^{d-1} . We write $\Delta_{0,d}$ instead of Δ_0 when we need to emphasize the dimension.

Lemma 4.2. *Let $\Delta_{0,d}$ be the Laplacian–Beltrami operator for \mathbb{S}^{d-1} . For $\xi \in \mathbb{S}^{d-1}$, write $\xi = (\sqrt{1-t^2}\eta, t)$ with $-1 \leq t \leq 1$ and $\eta \in \mathbb{S}^{d-2}$. Then*

$$(4.5) \quad \Delta_{0,d} = \frac{1}{(1-t^2)^{\frac{d-3}{2}}} \frac{\partial}{\partial t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} \right) + \frac{1}{1-t^2} \Delta_{0,d-1}.$$

Proof. We work with the expression for $\Delta_{0,d}$ in (4.3) and make a change of variables $(\xi_1, \dots, \xi_{d-1}) \mapsto (\eta_1, \dots, \eta_{d-2}, t)$ defined by

$$\xi_1 = \sqrt{1-t^2}\eta_1, \dots, \xi_{d-2} = \sqrt{1-t^2}\eta_{d-2}, \quad \xi_{d-1} = t,$$

where we have switched ξ_{d-1} and ξ_d for convenience. The chain rule gives

$$\frac{\partial}{\partial \xi_i} = \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \eta_i}, \quad 1 \leq i \leq d-2, \quad \text{and} \quad \frac{\partial}{\partial \xi_{d-1}} = \frac{t}{1-t^2} \sum_{j=1}^{d-2} \eta_j \frac{\partial}{\partial \eta_j} + \frac{\partial}{\partial t},$$

which can be used iteratively to compute $\Delta_{0,d}$ on writing (4.3) as

$$\begin{aligned} \Delta_{0,d} = (1-t^2) \frac{\partial^2}{\partial \xi_{d-1}^2} + \sum_{i=1}^{d-2} \frac{\partial^2}{\partial \xi_i^2} - 2t \sum_{i=1}^{d-2} \xi_i \frac{\partial^2}{\partial \xi_i \partial \xi_{d-1}} \\ - \sum_{i=1}^{d-2} \sum_{j=1}^{d-2} \xi_i \xi_j \frac{\partial^2}{\partial \xi_i \partial \xi_j} - (d-1) \sum_{i=1}^{d-1} \frac{\partial}{\partial \xi_i}. \end{aligned}$$

A straightforward computation and another use of (4.3) then leads to

$$\Delta_{0,d} = (1-t^2) \frac{\partial^2}{\partial t^2} - (d-1)t \frac{\partial}{\partial t} + \frac{1}{1-t^2} \Delta_{0,d-1},$$

which is precisely (4.5). \square

The formula (4.3) gives an explicit expression for Δ_0 in the local coordinates of \mathbb{S}^{d-1} . An explicit formula for Δ_0 in terms of spherical coordinates will be given in Section 5.

Let $\nabla = (\partial_1, \dots, \partial_d)$. The proof of Lemma 4.1 also shows that

$$(4.6) \quad \nabla = \frac{1}{r} \nabla_0 + \xi \frac{\partial}{\partial r}, \quad x = r\xi, \quad \xi \in \mathbb{S}^{d-1},$$

where ∇_0 is the spherical gradient, which is the spherical part of ∇ and involves only derivatives in ξ . Its explicit expression can be read off from (4.4). We shall not need this expression and will be content with the following expression.

Corollary 4.3. *Let $f \in C^2(\mathbb{S}^{d-1})$. Define $F(y) := f(y/\|y\|)$, $y \in \mathbb{R}^d$. Then*

$$(4.7) \quad \Delta_0 f(x) = \Delta F(x) \quad \text{and} \quad \nabla_0 f(x) = \nabla F(x), \quad x \in \mathbb{S}^{d-1}.$$

The corollary follows immediately from (4.2) and (4.6), since $x/\|x\|$ is independent of r . The expressions in (4.7) show that Δ_0 and ∇_0 are independent of the coordinates of \mathbb{S}^{d-1} . In fact, we could take (4.7) as the definition of Δ_0 and ∇_0 .

The usual Laplacian Δ can be expressed in terms of the dot product of ∇ , $\Delta = \nabla \cdot \nabla$, which can also be written—as is often in physics textbooks—as $\Delta = \nabla^2$. The analogue of this identity also holds on the sphere.

Lemma 4.4. *The Laplace–Beltrami operator satisfies*

$$(4.8) \quad \Delta_0 = \nabla_0 \cdot \nabla_0.$$

Proof. An application of (4.6) gives immediately

$$\Delta = \nabla \cdot \nabla = \frac{1}{r^2} \nabla_0 \cdot \nabla_0 + \frac{1}{r} \nabla_0 \left(\xi \frac{\partial}{\partial r} \right) + \xi \frac{\partial}{\partial r} \left(\frac{1}{r} \nabla_0 \right) + \frac{\partial^2}{\partial r^2}.$$

We note that $\xi \cdot \nabla_0 f(\xi) = 0$, since $\xi \in \mathbb{S}^{d-1}$ is in the normal direction of ξ whereas $\nabla_0 f(\xi)$, by (4.6), is on the tangent plane at ξ . Hence, we see that

$$\xi \frac{\partial}{\partial r} \left(\frac{1}{r} \nabla_0 \right) = -\frac{1}{r^2} \xi \cdot \nabla_0 + \frac{1}{r} \xi \cdot \nabla_0 \frac{\partial}{\partial r} = 0.$$

Using (4.6), a quick computation gives $\nabla_0 \cdot \xi = d - 1$, so that by the product rule,

$$\nabla_0 \left(\xi \frac{\partial}{\partial r} \right) = \nabla_0 \cdot \xi \frac{\partial}{\partial r} + \frac{\partial}{\partial r} \xi \cdot \nabla_0 = (d - 1) \frac{\partial}{\partial r}.$$

Consequently, we conclude that

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \nabla_0 \cdot \nabla_0.$$

Comparing this with (4.2) completes the proof. \square

Our next result shows that the spherical harmonics are eigenfunctions of the Laplace-Beltrami operator.

Theorem 4.5. *The spherical harmonics are eigenfunctions of Δ_0 ,*

$$(4.9) \quad \Delta_0 Y(\xi) = -n(n + d - 2)Y(\xi), \quad \forall Y \in \mathcal{H}_n^d, \quad \xi \in \mathbb{S}^{d-1}.$$

Proof. Let $x = r\xi$, $\xi \in \mathbb{S}^{d-1}$. Since $Y \in \mathcal{H}_n^d$ is homogeneous, $Y(x) = r^n Y(\xi)$ and by (4.2),

$$0 = \Delta Y(x) = n(n-1)r^{n-2}Y(\xi) + (d-1)nr^{n-2}Y(\xi) + r^{n-2}\Delta_0 Y(\xi),$$

which is (4.9) upon dividing by r^{n-2} . \square

The identity (4.9) also implies that Δ_0 is self-adjoint, which can also be proved directly and will be treated in the last section of this chapter, together with a number of other properties of the Laplace-Beltrami operator.

5. SPHERICAL HARMONICS IN SPHERICAL COORDINATES

The polar coordinates $(x_1, x_2) = (r \cos \theta, r \sin \theta)$, $r \geq 0$, $0 \leq \theta \leq 2\pi$, give coordinates for \mathbb{S}^1 when $r = 1$. The high-dimensional analogue is the spherical polar coordinates defined by

$$(5.1) \quad \begin{cases} x_1 = r \sin \theta_{d-1} \dots \sin \theta_2 \sin \theta_1, \\ x_2 = r \sin \theta_{d-1} \dots \sin \theta_2 \cos \theta_1, \\ \dots \\ x_{d-1} = r \sin \theta_{d-1} \cos \theta_{d-2}, \\ x_d = r \cos \theta_{d-1}, \end{cases}$$

where $r \geq 0$, $0 \leq \theta_1 \leq 2\pi$, $0 \leq \theta_i \leq \pi$ for $i = 2, \dots, d-1$. When $r = 1$ these are the coordinates for the unit sphere \mathbb{S}^{d-1} , and they are in fact defined recursively by

$$x = (\xi \sin \theta_{d-1}, \cos \theta_{d-1}) \in \mathbb{S}^{d-1}, \quad \xi \in \mathbb{S}^{d-2}.$$

Let $d\sigma = d\sigma_d$ be Lebesgue measure on \mathbb{S}^{d-1} . Then it is easy to verify that

$$(5.2) \quad d\sigma_d(x) = (\sin \theta_{d-1})^{d-2} d\theta_{d-1} d\sigma_{d-1}(\xi).$$

Since the Lebesgue measure of \mathbb{S}^1 is $d\theta_1$, it follows by induction that

$$(5.3) \quad d\sigma = d\sigma_d = \prod_{j=1}^{d-2} (\sin \theta_{d-j})^{d-j-1} d\theta_{d-1} \dots d\theta_2 d\theta_1$$

in the spherical coordinates (5.1). Furthermore, (5.2) shows that

$$(5.4) \quad \int_{\mathbb{S}^{d-1}} f(x) d\sigma_d(x) = \int_0^\pi \int_{\mathbb{S}^{d-2}} f(\xi \sin \theta, \cos \theta) d\sigma_{d-1}(\xi) (\sin \theta)^{d-2} d\theta.$$

The orthogonality (2.10) of the Gegenbauer polynomials can be written as

$$(5.5) \quad \int_0^\pi C_n^\lambda(\cos \theta) C_m^\lambda(\cos \theta) (\sin \theta)^{d-2} d\theta = \frac{\sqrt{\pi} \Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} h_n^\lambda \delta_{m,n}, \quad \lambda = \frac{d-2}{2}.$$

Together with (5.4), this allows us to write down a basis of spherical harmonics in terms of the Gegenbauer polynomials in the spherical coordinates.

Theorem 5.1. *For $d > 2$ and $\alpha \in \mathbb{N}_0^d$, define*

$$(5.6) \quad Y_\alpha(x) := [h_\alpha]^{-1} r^{|\alpha|} g_\alpha(\theta_1) \prod_{j=1}^{d-2} (\sin \theta_{d-j})^{|\alpha^{j+1}|} C_{\alpha_j}^{\lambda_j}(\cos \theta_{d-j}),$$

where $g_\alpha(\theta_1) = \cos \alpha_{d-1} \theta_1$ for $\alpha_d = 0$, $\sin \alpha_{d-1} \theta_1$ for $\alpha_d = 1$, $|\alpha^j| = \alpha_j + \dots + \alpha_{d-1}$, $\lambda_j = |\alpha^{j+1}| + (d-j-1)/2$, and

$$[h_\alpha]^2 := b_\alpha \prod_{j=1}^{d-2} \frac{\alpha_j! (\frac{d-j+1}{2})_{|\alpha^{j+1}|} (\alpha_j + \lambda_j)}{(2\lambda_j)_{\alpha_j} (\frac{d-j}{2})_{|\alpha^{j+1}|} \lambda_j}$$

in which $b_\alpha = 2$ if $\alpha_{d-1} + \alpha_d > 0$, while $b_\alpha = 1$ otherwise. Then $\{Y_\alpha : |\alpha| = n, \alpha_d = 0, 1\}$ is an orthonormal basis of \mathcal{H}_n^d ; that is, $\langle Y_\alpha, Y_\beta \rangle_{\mathbb{S}^{d-1}} = \delta_{\alpha,\beta}$.

Proof. To see that Y_α is a homogeneous polynomial, we use, by (5.1), the relation $\cos \theta_k = x_{k+1} / \sqrt{x_1^2 + \dots + x_{k+1}^2}$ for $1 \leq k \leq d-1$ to rewrite (5.6) as

$$Y_\alpha(x) = [h_\alpha]^{-1} g(x) \prod_{j=1}^{d-2} (x_1^2 + \dots + x_{d-j+1}^2)^{\alpha_j/2} C_{\alpha_j}^{\lambda_j} \left(\frac{x_{d-j+1}}{\sqrt{x_1^2 + \dots + x_{d-j+1}^2}} \right)$$

where $g(x) = \rho^{\alpha_{d-1}} \cos \alpha_{d-1} \theta_1$ for $\alpha_d = 0$, $\rho^{\alpha_{d-1}} \sin \alpha_{d-1} \theta_1$ for $\alpha_d = 1$, with $\rho = \sqrt{x_1^2 + x_2^2}$. Since $x_1 = \rho \sin \theta_1$ and $x_2 = \rho \cos \theta_1$ by (5.1), $g(x)$ is either the real part or the imaginary part of $(x_2 + ix_1)^{\alpha_{d-1}}$, which shows that it is a homogeneous polynomial of degree α_{d-1} in x . Since $C_n^\lambda(t)$ is even when n is even and odd when n is odd, we see that $Y_\alpha \in \mathcal{P}_n^d$. Using (5.4), we see that

$$\begin{aligned} \langle Y_\alpha, Y_{\alpha'} \rangle_{\mathbb{S}^{d-1}} &= \frac{h_\alpha^{-1} h_{\alpha'}^{-1}}{\omega_d} \int_0^{2\pi} g_\alpha(\theta_1) g_{\alpha'}(\theta_1) d\theta_1 \\ &\quad \times \prod_{j=1}^{d-2} \int_0^\pi C_{\alpha_j}^{\lambda_j}(\cos \theta_{d-j}) C_{\alpha'_j}^{\lambda_j}(\cos \theta_{d-j}) (\sin \theta_{d-j})^{2\lambda_j} d\theta_{d-j} \end{aligned}$$

from which the orthogonality follows from the orthogonality of the Gegenbauer polynomials (5.4) and that of $\cos m\theta$ and $\sin m\theta$ on $[0, 2\pi)$, and the formula for h_α follows from the normalization constant of the Gegenbauer polynomial. \square

For $d = 2$ and the polar coordinates $(x_1, x_2) = (r \cos \theta, r \sin \theta)$, it is easy to see that $\nabla_0 = \partial_\theta$, where $\partial_\theta = \partial/\partial\theta$. Hence by (4.8), the Laplace-Beltrami operator for $d = 2$ is $\Delta_0 = \partial_\theta^2$. Using (4.5) iteratively, we see that the Laplace-Beltrami operator Δ_0 has an explicit formula in the spherical coordinates (5.1),

$$(5.7) \quad \Delta_0 = \frac{1}{\sin^{d-2} \theta_{d-1}} \frac{\partial}{\partial \theta_{d-1}} \left[\sin^{d-2} \theta_{d-1} \frac{\partial}{\partial \theta_{d-1}} \right] \\ + \sum_{j=1}^{d-2} \frac{1}{\sin^2 \theta_{d-1} \cdots \sin^2 \theta_{j+1} \sin^{j-1} \theta_j} \frac{\partial}{\partial \theta_j} \left[\sin^{j-1} \theta_j \frac{\partial}{\partial \theta_j} \right].$$

6. SPHERICAL HARMONICS IN TWO AND THREE VARIABLES

Since spherical harmonics in two and three variables are used most often in applications, we state their properties in this section.

6.1. Spherical harmonics in two variables. For $d = 2$, $\dim \mathcal{H}_n^2 = 2$. An orthogonal basis of \mathcal{H}_n^2 is given by the real and imaginary parts of $(x_1 + ix_2)^n$, since both are homogeneous of degree n and are harmonic as the real and imaginary parts of an analytic function. In polar coordinates $(x_1, x_2) = (r \cos \theta, r \sin \theta)$ of \mathbb{R}^2 , this basis is given by

$$(6.1) \quad Y_n^{(1)}(x) = r^n \cos n\theta, \quad Y_n^{(2)}(x) = r^n \sin n\theta.$$

Hence, restricting to the circle \mathbb{S}^1 , the spherical harmonics are precisely the cosine and sine functions. In particular, spherical harmonic expansions on \mathbb{S}^1 are the classical Fourier expansions in cosine and sine functions.

As homogeneous polynomials, the basis (6.1) is given explicitly in terms of the Chebyshev polynomials T_n and U_n defined by

$$T_n(t) = \cos n\theta \quad \text{and} \quad U_n(t) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad \text{where } t = \cos \theta,$$

which are related to the Gegenbauer polynomials: $U_n(t) = C_n^1(t)$ and

$$\lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} C_n^\lambda(x) = \frac{2}{n} T_n(x).$$

The basis in (6.1) can be rewritten then as

$$(6.2) \quad Y_n^{(1)}(x) = r^n T_n\left(\frac{x_1}{r}\right), \quad Y_n^{(2)}(x) = r^{n-1} x_2 U_{n-1}\left(\frac{x_1}{r}\right),$$

which shows explicitly that these are homogeneous polynomials since $r = \sqrt{x_1^2 + x_2^2}$ and both $T_n(t)$ and $U_n(t)$ are even if n is even, odd if n is odd.

When $d = 2$, the zonal polynomial is given by $T_n(\langle x, y \rangle) = \cos n(\theta - \phi)$ and the addition formula (2.8) becomes, by (6.1), the addition formula

$$\cos n\theta \cos n\phi + \sin n\theta \sin n\phi = \cos n(\theta - \phi).$$

The expression (4.2) of the Laplace operator in polar coordinates becomes

$$\Delta = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \frac{d^2}{d\theta^2},$$

in particular, the Laplace-Beltrami operator on \mathbb{S}^1 is simply $\Delta_0 = d^2/d\theta^2$.

6.2. Spherical harmonics in three variables. The space \mathcal{H}_n^3 of spherical harmonics of degree n has dimension $2n + 1$. For $d = 3$ the spherical polar coordinates (5.1) are written as

$$(6.3) \quad \begin{cases} x_1 = r \sin \theta \sin \phi, \\ x_2 = r \sin \theta \cos \phi, \\ x_3 = r \cos \theta, \end{cases} \quad , \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad r > 0.$$

The surface area of \mathbb{S}^2 is 4π and the integral over \mathbb{S}^{d-1} is parameterized by

$$(6.4) \quad \int_{\mathbb{S}^2} f(x) d\sigma = \int_0^\pi \int_0^{2\pi} f(\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta) d\phi \sin \theta d\theta.$$

The orthogonal basis (5.6) in spherical coordinates becomes

$$(6.5) \quad \begin{aligned} Y_{k,1}^n(\theta, \phi) &= (\sin \theta)^k C_{n-k}^{k+\frac{1}{2}}(\cos \theta) \cos k\phi, \quad 0 \leq k \leq n, \\ Y_{k,2}^n(\theta, \phi) &= (\sin \theta)^k C_{n-k}^{k+\frac{1}{2}}(\cos \theta) \sin k\phi, \quad 1 \leq k \leq n. \end{aligned}$$

Their $L^2(\mathbb{S}^2)$ norms can be deduced from (5.6). This basis is often written in terms of the associated Legendre polynomials $P_n^k(t)$ defined by

$$P_n^k(x) := (-1)^n (1-x^2)^{k/2} \frac{d^k}{dx^k} P_n(x) = (2k-1)!! (-1)^n (1-x^2)^{k/2} C_{n-k}^{k+1/2}(x),$$

where $P_n(t) = C_n^0(t)$ denotes the Legendre polynomial of degree n (see Appendix B for properties of P_n and P_n^k), and in terms of $\{e^{ik\phi}, e^{-ik\phi}\}$ instead of $\{\cos k\phi, \sin k\phi\}$. In this way an orthonormal basis of \mathcal{H}_n^3 is given by

$$(6.6) \quad Y_{k,n}(\theta, \phi) = \left(\frac{(2n+1)(n-k)!}{(n+k)!} \right)^{1/2} P_n^k(\cos \theta) e^{ik\phi}, \quad -n \leq k \leq n.$$

The addition formula (2.8) then reads, assuming that x and y have spherical coordinates (θ, ϕ) and (θ', ϕ') , respectively,

$$(6.7) \quad \sum_{k=-n}^n Y_{k,n}(\theta, \phi) Y_{k,n}(\theta', \phi') = (2n+1) P_n(\langle x, y \rangle).$$

In terms of the coordinates (6.3), the Laplace–Beltrami operator is given by

$$(6.8) \quad \Delta_0 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

as seen from (5.7).

7. REPRESENTATION OF THE ROTATION GROUP

In this section we show that the representation of the group $SO(d)$ in spaces of harmonic polynomials is irreducible.

A representation of $SO(d)$ is a homomorphism from $SO(d)$ to the group of nonsingular continuous linear transformations of $L^2(\mathbb{S}^{d-1})$. We associate with each element $Q \in SO(d)$ an operator $T(Q)$ in the space of $L^2(\mathbb{S}^{d-1})$, defined by

$$(7.1) \quad T(Q)f(x) = f(Q^{-1}x), \quad x \in \mathbb{S}^{d-1}.$$

Evidently, for each $Q \in SO(d)$, $T(Q)$ is a nonsingular linear transformation of $L^2(\mathbb{S}^{d-1})$ and T is a homomorphism,

$$T(Q_1 Q_2) = T(Q_1) T(Q_2), \quad \forall Q_1, Q_2 \in SO(d).$$

Thus, T is a representation of $SO(d)$. Since $d\sigma$ is invariant under rotations, $\|T(Q)f\|_2 = \|f\|_2$ in the $L^2(\mathbb{S}^{d-1})$ norm, so that $T(Q)$ is unitary.

A linear space \mathcal{U} is called invariant under T if $T(Q)$ maps \mathcal{U} to itself for all $Q \in SO(d)$. The null space and $L^2(\mathbb{S}^{d-1})$ itself are trivial invariant subspaces. A representation T is called irreducible if it has only trivial invariant subspaces. The space \mathcal{H}_n^d is an invariant subspace of T in (7.1).

Let $T_{n,d}$ denote the representation of $SO(d)$ corresponding to T in the invariant subspace \mathcal{H}_n^d . We want to show that $T_{n,d}$ is irreducible.

Lemma 7.1. *A spherical harmonic $Y \in \mathcal{H}_n^d$ is invariant under all rotations in $SO(d)$ that leaves x_d fixed if and only if*

$$(7.2) \quad Y(x) = c\|x\|^n C_n^\lambda\left(\frac{x_d}{\|x\|}\right), \quad \lambda = \frac{d-2}{2}.$$

where c is a constant.

Proof. If Y is invariant under rotations that fix x_d and is a homogeneous polynomial of degree n , then it can be written as

$$Y(x) = \sum_{0 \leq j \leq n/2} b_j x_d^{n-2j} (x_1^2 + \dots + x_{d-1}^2)^j = \sum_{0 \leq j \leq n/2} c_j x_d^{n-2j} \|x\|^{2j}.$$

where the second equal sign follows from expanding $(\|x\|^2 - x_d^2)^j$ and changing the order of summation. Since Y is harmonic, computing $\Delta Y(x) = 0$ shows that c_j satisfies the recurrence relation

$$4(j+1)(n-j-1)c_{j+1} + (n-2j)(n-2j-1)c_j = 0.$$

Solving the recurrence equation for c_j we conclude that

$$Y(x) = c_0 \sum_{0 \leq j \leq n/2} \frac{(-\frac{n}{2})_j (\frac{1-n}{2})_j}{j! (1-n-d-2)_j} x_d^{n-2j} \|x\|^{2j}.$$

Consequently, (7.2) follows from the formula (??) for Gegenbauer polynomials. Since the function Y in (7.2) is clearly invariant under all rotations that fix x_d and we have just shown that it is harmonic, the proof is complete. \square

Theorem 7.2. *The representation $T_{n,d}$ of $SO(d)$ on \mathcal{H}_n^d is irreducible.*

Proof. Assume that \mathcal{U} is an invariant subspace of \mathcal{H}_n^d and \mathcal{U} is not a null space. Let $\{Y_j : 1 \leq j \leq M\}$, $M \leq \dim \mathcal{H}_n^d$, be an orthonormal basis of \mathcal{U} . Following the proof of Lemma 2.5, there is a polynomial $F(t)$ of one variable such that $\sum_{j=1}^M Y_j(x)Y_j(y) = F(\langle x, y \rangle)$. In particular, setting $y = e_d = (0, \dots, 0, 1)$ shows that $F(\langle x, e_d \rangle)$ is in \mathcal{H}_n^d and it is evidently invariant under rotations in $SO(d)$ that fix x_d . Hence, by Lemma 7.1 $F(\langle x, e_d \rangle) = c\|x\|^n C_n^\lambda(\frac{x_d}{\|x\|})$. In particular, this shows that $\|x\|^n C_n^\lambda(\frac{x_d}{\|x\|}) \in \mathcal{U}$. On the other hand, let \mathcal{U}^\perp denote the orthogonal complement of \mathcal{U} in \mathcal{H}_n^d . If $f \in \mathcal{U}^\perp$ and $g \in \mathcal{U}$, then $\langle T(Q)f, g \rangle_{\mathbb{S}^{d-1}} = \langle f, T(Q^{-1})g \rangle_{\mathbb{S}^{d-1}} = 0$, which shows that \mathcal{U}^\perp is also an invariant subspace of \mathcal{H}_n^d . Applying the same argument as for \mathcal{U} shows then $\|x\|^n C_n^\lambda(\frac{x_d}{\|x\|}) \in \mathcal{U}^\perp$, which contradicts to $\mathcal{U} \cap \mathcal{U}^\perp = \{0\}$. Thus, \mathcal{U} must be trivial. \square

8. ANGULAR DERIVATIVES AND THE LAPLACE–BELTRAMI OPERATOR

Consider the case $d = 2$ and the polar coordinates $(x_1, x_2) = (r \cos \theta, r \sin \theta)$. Let $\partial_r = \partial/\partial r$ and $\partial_\theta = \partial/\partial \theta$, while we retain ∂_1 and ∂_2 for the partial derivatives with respect to x_1 and x_2 . It then follows that

$$\partial_1 = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta, \quad \partial_2 = \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta.$$

From these relations it follows easily that the angular derivative ∂_θ can also be written as $\partial_\theta = x_1 \partial_2 - x_2 \partial_1$ and the operator Δ_0 is $\Delta_0 = \partial_\theta^2$. We introduce angular derivatives in higher dimensions as follows.

Definition 8.1. For $x \in \mathbb{R}^d$ and $1 \leq i \neq j \leq d$, define

$$(8.1) \quad D_{i,j} := x_i \partial_j - x_j \partial_i = \frac{\partial}{\partial \theta_{i,j}},$$

where $\theta_{i,j}$ is the angle of polar coordinates in the (x_i, x_j) -plane, defined by $(x_i, x_j) = r_{i,j}(\cos \theta_{i,j}, \sin \theta_{i,j})$, $r_{i,j} \geq 0$ and $0 \leq \theta_{i,j} \leq 2\pi$.

By its definition with partial derivatives on \mathbb{R}^d , $D_{i,j}$ acts on \mathbb{R}^d , yet the second equality in (8.1) shows that it acts on the sphere \mathbb{S}^{d-1} . Thus, for f defined on \mathbb{R}^d ,

$$(8.2) \quad (D_{i,j}f)(\xi) = D_{i,j}[f(\xi)], \quad \xi \in \mathbb{S}^{d-1},$$

where the right-hand side means that $D_{i,j}$ is acting on $f(\xi)$.

Since $D_{j,i} = -D_{i,j}$, the number of distinct operators $D_{i,j}$ is $\binom{d}{2}$. The operator Δ_0 can be decomposed in terms of them.

Theorem 8.2. On \mathbb{S}^{d-1} , Δ_0 satisfies the decomposition

$$(8.3) \quad \Delta_0 = \sum_{1 \leq i < j \leq d} D_{i,j}^2.$$

Proof. Let $F(x) = f\left(\frac{x}{\|x\|}\right)$. A straightforward computation shows that

$$\sum_{1 \leq i < j \leq d} D_{i,j}^2 F(x) = (\Delta F)(x) - \frac{1}{\|x\|^2} \sum_{i=1}^d \sum_{j=1}^d x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} - \frac{d-1}{\|x\|} \sum_{i=1}^d \frac{\partial F}{\partial x_i}.$$

Consequently, restricting to \mathbb{S}^{d-1} and comparing with (4.3), (8.3) follows. \square

Let $Q_{i,j,\theta}$ denote a rotation by the angle θ in the (x_i, x_j) -plane, oriented so that $(x_i, x_j) = (s \cos \theta, s \sin \theta)$. Then $T(Q_{i,j,\theta})$, defined in (7.1), maps f into $T(Q_{i,j,\theta})f(x) = f(Q_{i,j,-\theta}x)$. Written explicitly, for example, for $(i, j) = (1, 2)$, we have

$$(8.4) \quad T(Q_{1,2,\theta}f)(x) = f(x_1 \cos \theta + x_2 \sin \theta, -x_1 \sin \theta + x_2 \cos \theta, x_3, \dots, x_d).$$

Then $D_{i,j}$ is the infinitesimal operator of $T(Q)$,

$$(8.5) \quad \left. \frac{dT(Q_{i,j,\theta})}{d\theta} \right|_{\theta=0} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} = D_{i,j},$$

where the first equality follows from (8.4). The infinitesimal operator plays an important role in representation theory; see, for example, [12].

The operators $D_{i,j}$ will play an important role for approximation theory on the sphere. We state several more properties of these operators.

Lemma 8.3. *For $1 \leq i < j \leq d$, the operators $D_{i,j}$ commute with Δ_0 . In particular, $D_{i,j}$ maps \mathcal{H}_n^d to itself.*

Proof. By symmetry, we only to show only that $D_{1,2}$. Let $[A, B] = AB - BA$ denote the commutator of A and B . A quick computation shows that

$$[\partial_j, D_{k,l}] = \delta_{j,k} \partial_l - \delta_{j,l} \partial_k, \quad [x_i, D_{k,l}] = \delta_{i,k} x_l - \delta_{i,l} x_k,$$

from which it is easy to see that, for $1 \leq i < j \leq d$, $1 \leq k < l \leq d$, we have

$$(8.6) \quad [D_{i,j}, D_{k,l}] = -\delta_{i,k} D_{j,l} + \delta_{i,l} D_{j,k} + \delta_{j,k} D_{i,l} - \delta_{j,l} D_{i,k}.$$

Using (8.6), a simple computation shows that $[D_{1,2}, D_{1,l}^2] = -(D_{1,l} D_{2,l} + D_{2,l} D_{1,l})$ and $[D_{1,2}, D_{2,l}^2] = D_{1,l} D_{2,l} + D_{2,l} D_{1,l}$, so that $[D_{1,2}, D_{1,l}^2 + D_{2,l}^2] = 0$ for $l \geq 2$. Moreover, by (8.6), $[D_{1,2}, D_{k,l}^2] = 0$ whenever $3 \leq k < l \leq d$. Summing over (k, l) for $1 \leq k < l \leq d$ then proves $[D_{1,2}, \Delta_0] = 0$. \square

Proposition 8.4. *For $f, g \in C^1(\mathbb{S}^{d-1})$ and $1 \leq i \neq j \leq d$,*

$$(8.7) \quad \int_{\mathbb{S}^{d-1}} f(x) D_{i,j} g(x) d\sigma(x) = - \int_{\mathbb{S}^{d-1}} D_{i,j} f(x) g(x) d\sigma(x).$$

Proof. By the rotation invariance of the Lebesgue measure $d\sigma$, we obtain, for every $\theta \in [-\pi, \pi]$,

$$\int_{\mathbb{S}^{d-1}} f(x) g(Q_{i,j,-\theta} x) d\sigma(x) = \int_{\mathbb{S}^{d-1}} f(Q_{i,j,\theta} x) g(x) d\sigma(x).$$

Differentiating both sides of this identity with respect to θ and evaluating the resulting equation at $\theta = 0$ leads to, by (8.5), the desired equation (8.7). \square

The equation (8.7) allows us to define distributional derivatives $D_{i,j}^r$ on \mathbb{S}^{d-1} for $r \in \mathbb{N}$ via the identity, with $g \in C^\infty(\mathbb{S}^{d-1})$,

$$(8.8) \quad \int_{\mathbb{S}^{d-1}} D_{i,j}^r f(x) g(x) d\sigma(x) = (-1)^r \int_{\mathbb{S}^{d-1}} f(x) D_{i,j}^r g(x) d\sigma(x).$$

Summing (8.8) with $r = 2$ over $i < j$ and applying (8.3), it follows immediately that Δ_0 is self-adjoint, which can also be deduced from Theorem 4.5.

Corollary 8.5. *For $f, g \in C^2(\mathbb{S}^{d-1})$,*

$$\int_{\mathbb{S}^{d-1}} f(x) \Delta_0 g(x) d\sigma = \int_{\mathbb{S}^{d-1}} \Delta_0 f(x) g(x) d\sigma.$$

The spherical gradient ∇_0 is a vector of first order differential operator on the sphere, which can be written in terms of $D_{i,j}$ as follows.

Lemma 8.6. *For $f \in C^1(\mathbb{S}^{d-1})$ and $1 \leq j \leq d$, the j -th component of $\nabla_0 f$ satisfies*

$$(8.9) \quad (\nabla_0)_j f(\xi) = \sum_{1 \leq i \leq d, i \neq j} \xi_i D_{i,j} f(\xi), \quad \xi \in \mathbb{S}^{d-1}.$$

Furthermore, for $f, g \in C^1(\mathbb{S}^{d-1})$, the following identity holds:

$$(8.10) \quad \nabla_0 f(\xi) \cdot \nabla_0 g(\xi) = \sum_{1 \leq i < j \leq d} D_{i,j} f(\xi) D_{i,j} g(\xi), \quad \xi \in \mathbb{S}^{d-1}.$$

Proof. Let $F(y) = f(y/\|y\|)$. From the definition and $\|\xi\| = 1$, we obtain

$$(8.11) \quad \begin{aligned} (\nabla_0)_j f(\xi) &= \frac{\partial}{\partial x_j} F(\xi) = \partial_j f - \xi_j \sum_{i=1}^d \xi_i \partial_i f \\ &= \partial_j f \sum_{i=1}^d \xi_i^2 - \xi_j \sum_{i=1}^d \xi_i \partial_i f = \sum_{i=1}^d \xi_i D_{i,j} f, \end{aligned}$$

which gives (8.9) since $D_{i,i} f = 0$. (8.11) means that

$$(8.12) \quad \nabla_0 f(\xi) = \nabla f(\xi) - \xi(\xi \cdot \nabla f).$$

Since $\xi \cdot \nabla_0 f(\xi) = 0$, using (8.11) and the definition of $D_{i,j}$, it follows that

$$\nabla_0 f \cdot \nabla_0 g = \nabla_0 f \cdot \nabla g = \sum_{j=1}^d \partial_j f \partial_j g - \sum_{j=1}^d \sum_{i=1}^d \xi_i \xi_j \partial_i f \partial_j g = \sum_{i < j} D_{i,j} f(\xi) D_{i,j} g(\xi),$$

where the second equality uses $\|\xi\| = 1$. \square

As an application, we state an integration by parts formula on the sphere.

Proposition 8.7. *For $f, g \in C^1(\mathbb{S}^{d-1})$,*

$$(8.13) \quad \int_{\mathbb{S}^{d-1}} f(x) \nabla_0 g(x) d\sigma = - \int_{\mathbb{S}^{d-1}} (\nabla_0 f(x) - (d-1)x f(x)) g(x) d\sigma.$$

Furthermore, for $f \in C^2(\mathbb{S}^{d-1})$ and $g \in C^1(\mathbb{S}^{d-1})$,

$$(8.14) \quad \int_{\mathbb{S}^{d-1}} \nabla_0 f \cdot \nabla_0 g d\sigma = - \int_{\mathbb{S}^{d-1}} \Delta_0 f(x) g(x) d\sigma.$$

Proof. Using the expression (8.11) and applying (8.7), we obtain

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} f(x) (\nabla_0)_j g(x) d\sigma &= \sum_{1 \leq i \leq d, i \neq j} \int_{\mathbb{S}^{d-1}} x_i f(x) D_{i,j} g(x) d\sigma \\ &= - \sum_{1 \leq i \leq d, i \neq j} \int_{\mathbb{S}^{d-1}} D_{i,j} (x_i f(x)) g(x) d\sigma. \end{aligned}$$

By the chain rule, recalling (8.2) if necessary, we have $D_{i,j}(x_i f(x)) = x_i D_{i,j} f - x_j f(x)$. Hence we obtain

$$\int_{\mathbb{S}^{d-1}} f(x) (\nabla_0)_j g(x) d\sigma = - \int_{\mathbb{S}^{d-1}} \left(\sum_{1 \leq i \leq d, i \neq j} x_i D_{i,j} f(x) - (d-1)f(x) \right) g(x) d\sigma,$$

which gives the j -th component of (8.13). Equation (8.14) follows immediately from (8.10), (8.7) and (8.3). \square

9. NOTES

Spherical harmonics appear in many disciplines and in many different branches of mathematics. Many books contain parts of the theory of spherical harmonics. Our treatment covers what is needed for harmonic analysis and approximation theory in this book. Below we comment on some books that we have consulted.

A classical treatise on spherical harmonics is [8], a good source for classical results. A short but nice expository is [9], which was later expanded into [10]. The reference book [5] contains a chapter on spherical harmonics. A rich resource for spherical harmonics in Fourier analysis is [11]. Applications to and connections

with group representation are studied extensively in [12]; see also [7]. For their role in the context of orthogonal polynomials of several variables, see [4] as well as [5]. The book [1] contains a chapter on spherical harmonics in light of special functions. The theory of harmonic functions is treated in [3], including material on spherical harmonics. The book [6] deals with spherical harmonics in geometric applications. Finally, the recent book [2] provides an introduction to spherical harmonics and approximation on the sphere from the perspective of applications in numerical analysis.

Aside from their role in representation theory, the operators $D_{i,j}$ do not seem to have received much attention in analysis. Most of the materials in Section 8 have not previously appeared in books. These operators play an important role in our development of approximation theory on the sphere.

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Approximation Theory and Harmonic Analysis on Spheres and Balls

F. Dai and Y. Xu

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DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA, , EDMONTON, ALBERTA T6G 2G1, CANADA.

E-mail address: `dfeng@math.ualberta.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403-1222.

E-mail address: `yuan@math.uoregon.edu`